数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成２１年８月２５日（火）　１０：００〜１３：００

５問出題，３問解答

◦ This booklet is an informal English translation of the original examination booklet.

- **Answer three problems out of Problem 1 ∼ Problem 5.**
- **Answer in Japanese or English.**

Problem 1

(1) Show that for a positive-definite symmetric real matrix A there uniquely exists a positive-definite symmetric real matrix B such that $A = B^2$.

(2) Show that for a nonsingular real matrix F there uniquely exist an orthogonal matrix R and a positive-definite symmetric real matrix U such that $F = RU$.

(3) Show that for a nonsingular real matrix F there uniquely exist an orthogonal matrix R and positive-definite symmetric real matrices U and V such that $F = RU = VR$.

Problem 2

Each item produced in a factory is defective with probability ε $(0 < \varepsilon < 1)$. Let X be a random variable such that $X = 0$ if the item is non-defective and $X = 1$ if it is defective.

Answer the following questions (1), (2) and (3).

(1) Each produced item is inspected before shipment. Let Y be a random variable such that $Y = 0$ if the inspection result indicates that the item is non-defective and $Y = 1$ if the inspection result indicates that the item is defective. When $X = 0$, we have $Y = 0$ with probability 0.9 and $Y = 1$ with probability 0.1. When $X = 1$, we have $Y = 0$ with probability 0.1 and $Y = 1$ with probability 0.9. The item is shipped if $Y = 0$, and it is discarded if $Y = 1$. Obtain the probability $\mathrm{P}(X = 1|Y = 0)$ that an item shipped from the factory is defective.

(2) The same inspection as in (1) is repeated n times before the produced item is shipped. Let Z_n be a random variable to represent the number of inspection results indicating that the item is defective. Here the n inspection results are independent of each other. Show that the conditional probability $\mathrm{P}(X = 1|Z_n = z_n)$ $(z_n = 0, 1, \ldots, n)$ is a function of ε and $n - 2z_n$, the latter of which is equal to the difference between the number $n - z_n$ of inspection results indicating that the item is non-defective and the number z_n of inspection results indicating that the item is defective.

(3) Let $q = \mathrm{P}(X = 1|Z_2 = 0)$ and $r = \mathrm{P}(X = 1|Z_2 = 2)$. The same inspection as in (1) is repeated until the conditional probability $\mathrm{P}(X = 1|Z_n = z_n)$ based on the inspection result $Z_n = z_n$ first becomes at most q or at least r. Obtain the expected number of inspections performed before termination when the item is non-defective.

Problem 3

(1) Suppose that A is a bounded set† in an n-dimensional Euclidean space and that the volume $v(A)$ of A is greater than 1 (i.e., $v(A) > 1$). We consider the intersection $(A + \boldsymbol{g}) \cap C$ of

$$A + \boldsymbol{g} = \{\boldsymbol{x} + \boldsymbol{g} \mid \boldsymbol{x} \in A\},$$

which denotes the set obtained by translating A by an n-dimensional integral vector $\boldsymbol{g}$, and

$$C = \{\boldsymbol{x} \mid \boldsymbol{x} = (x_1, \ldots, x_n),\ 0 \leq x_i < 1\ (i = 1, \ldots, n)\}.$$

Prove

$$\sum_{\boldsymbol{g} \in \mathbb{Z}^n} v\big((A + \boldsymbol{g}) \cap C\big) = v(A),$$

where $\mathbb{Z}^n$ denotes the set of all n-dimensional integral vectors.

(2) Prove that there exist two distinct integral vectors $\boldsymbol{g}$ and $\boldsymbol{h}$ such that the intersection of $(A + \boldsymbol{g}) \cap C$ and $(A + \boldsymbol{h}) \cap C$ is nonempty.

(3) Prove that there exist two distinct points $\boldsymbol{x}$ and $\boldsymbol{y}$ in A such that $\boldsymbol{x} - \boldsymbol{y} \in \mathbb{Z}^n$.

(4) Suppose that B is a bounded convex set‡ which is symmetric with respect to the origin in the n-dimensional Euclidean space and satisfies $v(B) > 2^n$. Prove that B contains a nonzero integral vector by applying the result of (3) to the set $\frac{1}{2}B = \{\frac{1}{2}\boldsymbol{x} \mid \boldsymbol{x} \in B\}$.

(5) Let $R = (r_{ij})$ be a nonsingular real matrix of order 3, and denote the determinant of R by $\det R$. Prove that for any α satisfying the inequality $\alpha > \sqrt[3]{3!|\det R|}$ there exists a nonzero integral vector $\boldsymbol{g} = (g_1, g_2, g_3) \in \mathbb{Z}^3$ such that

$$\sum_{i=1}^{3} \Big|\sum_{j=1}^{3} r_{ij} g_j\Big| < \alpha.$$

† Strictly speaking, A is a bounded measurable set, and $v(A)$ is the measure of A.

‡ B is said to be symmetric with respect to the origin, if $\boldsymbol{x} \in B$ implies $-\boldsymbol{x} \in B$. B is said to be convex if, for any two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in B, it contains the line segment joining $\boldsymbol{x}$ and $\boldsymbol{y}$. In particular, for any two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in B, we have $\frac{1}{2}(\boldsymbol{x} + \boldsymbol{y}) \in B$. It is known that a bounded convex set is measurable.

Problem 4

Let us consider a system consisting of three water-tanks 1, 2, and 3 (Fig. 1).

Water in the three tanks constantly flows out at the rates of r_1, r_2, and r_3 per unit time, respectively. At the same time, water is supplied to one of the three tanks at the rate of $R = r_1 + r_2 + r_3$ per unit time. When a tank becomes empty, the water supply is switched to the tank. Assume that the system stops if two tanks simultaneously become empty. Water does not overflow from the tanks.

Let $q(t) \in \{1, 2, 3\}$ denote the tank to which water is supplied at time t. Then, the amount of water $w_p(t)$ in tank p at time t obeys the following differential equation:

$$\frac{\mathrm{d}w_p(t)}{\mathrm{d}t} = R\delta_{pq(t)} - r_p \qquad (p = 1, 2, 3),$$

where δ is the Kronecker delta ($\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \neq q$).

In the following, assume that $r_1 = r_2 = r_3 = \dfrac{1}{3}$ and that the total amount of water in the three tanks $w_1(t) + w_2(t) + w_3(t)$, which is constant, is equal to 1.

(1) Let t_i denote the time when one or two tanks become empty for the ith time after time 0. If two tanks simultaneously become empty and the system stops at time t_i, the subsequent sequence is defined by $t_i = t_{i+1} = t_{i+2} = \cdots$.

Suppose that tank 1 becomes empty at time t_i. Let $\alpha = w_3(t_i)$ be the amount of water in tank 3 at time t_i. Express the amounts of water in the three tanks $w_1(t_{i+1})$, $w_2(t_{i+1})$, and $w_3(t_{i+1})$ at time t_{i+1} using α.

(2) Since one or two tanks are empty at time t_i, the state of the system at that moment can be described by a point x_i in the half-open interval $[0, 1) = \{x \mid 0 \leq x < 1\}$ as follows:

$$x_i = \begin{cases} \dfrac{w_3(t_i)}{3} & (\text{if } w_1(t_i) = 0 \text{ and } w_2(t_i) \neq 0), \\ \dfrac{w_1(t_i) + 1}{3} & (\text{if } w_2(t_i) = 0 \text{ and } w_3(t_i) \neq 0), \\ \dfrac{w_2(t_i) + 2}{3} & (\text{if } w_3(t_i) = 0 \text{ and } w_1(t_i) \neq 0). \end{cases}$$

Then, x_{i+1} is uniquely determined from x_i, and can be written in the form of $x_{i+1} = f(x_i)$. Obtain the map $f\colon [0, 1) \to [0, 1)$. Draw the graph of f.

(3) Show that for any two distinct points x and $y \in [0, 1)$, there exists an integer $k \geq 0$ such that $\left|f^k(y) - f^k(x)\right| \geq \dfrac{1}{4}$. Here $f^0(x) = x$ and $f^k(x) = f(f^{k-1}(x))$ ($k = 1, 2, \ldots$).

[Hint] First show that, if $|y - x| < \dfrac{1}{4}$, then $|f(y) - f(x)| \geq 2|y - x|$.

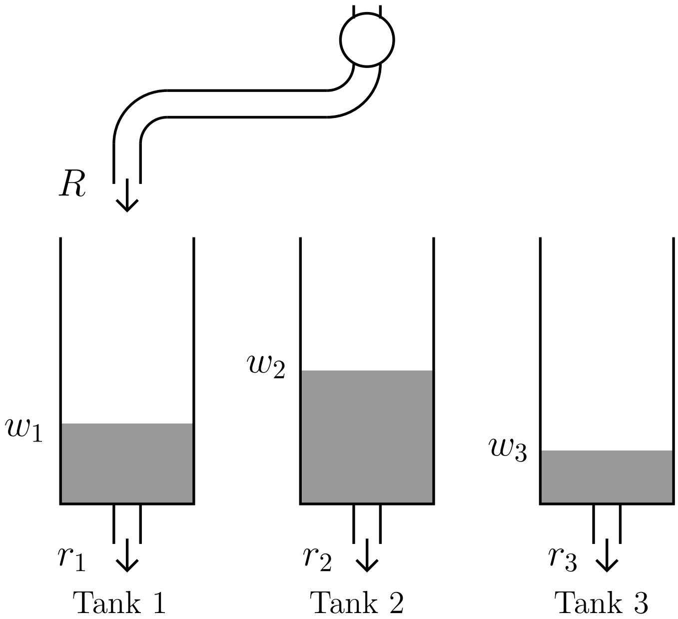

Fig. 1

Problem 5

(1) Given $2n+1$ positive integers $p_1, p_2, \ldots, p_n, s_1, s_2, \ldots, s_n, S$, let us consider solving the following optimization problem by the dynamic programming method.

$$\begin{aligned} \text{(Problem P)} \qquad & \max \quad \sum_{i=1}^{n} p_i x_i & \\ & \text{subject to} \quad \sum_{i=1}^{n} s_i x_i \leq S & (*) \\ & \qquad x_i \in \{0,1\} \quad (i = 1, \ldots, n). & (**) \end{aligned}$$

In order to solve Problem P, for $j = 1, \ldots, n$ and $s = 0, 1, \ldots, S$, we consider the optimization problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{j} p_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{j} s_i x_i = s \\ & x_i \in \{0,1\} \quad (i = 1, \ldots, j), \end{aligned}$$

and denote by $A(j,s)$ the optimal value (i.e., the maximum value of the objective function) of this problem. Here we define $A(j,s) = -\infty$ if the problem has no feasible solution. Then we can solve Problem P by letting $A(0,0) = 0$ and $A(0,s) = -\infty$ $(s \neq 0)$, and by using the following recurrence for $A(j,s)$ $(j = 1, \ldots, n)$:

$$\begin{aligned} A(j,s) &= A(j-1,s), \quad \text{if } s < s_j, \\ A(j,s) &= \max\{A(j-1,s), p_j + A(j-1, s-s_j)\}, \quad \text{otherwise.} \end{aligned}$$

Compute the optimal value for the problem instance with $n = 5$, $S = 5$, $p_1 = 2$, $p_2 = 3$, $p_3 = 2$, $p_4 = 1$, $p_5 = 3$, $s_1 = 2$, $s_2 = 3$, $s_3 = 1$, $s_4 = 2$, $s_5 = 1$ by applying the dynamic programming method based on the recurrence above. Give all the values $A(j,s)$ which are needed to compute the optimal value.

(2) Give an algorithm based on the recurrence in (1). Show the time complexity of the algorithm and explain whether it is polynomial in the input length.

(3) Consider Problem Q that is obtained from Problem P by replacing constraint $(**)$ with $x_i \in \{0, 1, \ldots, 10\}$ $(i = 1, \ldots, n)$. Obtain a recurrence as in (1).

(4) Consider Problem R that is obtained from Problem P by adding a new constraint $\sum_{i=1}^{n} w_i x_i \leq W$, where $w_1, \ldots, w_n$, and W are all positive integers. Obtain a recurrence as in (1).