数理情報学専攻

修士課程入学試験問題

専門科目 数理情報学

平成１６年8月２４日（火）　１０：００～１３：００

５問出題，３問解答

○ This booklet is an informal English translation of the original examination booklet.

- **Answer three problems out of Problem 1 ∼ Problem 5.**
- **Answer in Japanese or English.**
- **Do not take this booklet with you, but leave it on the desk.**

Examinee's number	No.

Problem 1

Consider a system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & y_3/I_3 & -y_2/I_2 \\ -y_3/I_3 & 0 & y_1/I_1 \\ y_2/I_2 & -y_1/I_1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad (*)$$

where y_1, y_2, and y_3 are functions of time t, and I_1, I_2, and I_3 are positive constants.

(1) Show that the system $(*)$ has the following two quantities as invariants:

$$L = {y_1}^2 + {y_2}^2 + {y_3}^2, \qquad K = \frac{1}{2}\left(\frac{{y_1}^2}{I_1} + \frac{{y_2}^2}{I_2} + \frac{{y_3}^2}{I_3}\right).$$

Generally, for a system of differential equations

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{z}), \qquad (**)$$

where $\boldsymbol{z} = (z_1, z_2, \ldots, z_m)^\top$ is a vector of functions of time t, the system of difference equations

$$\frac{\boldsymbol{z}^{(n+1)} - \boldsymbol{z}^{(n)}}{h} = \boldsymbol{f}\left(\frac{\boldsymbol{z}^{(n+1)} + \boldsymbol{z}^{(n)}}{2}\right)$$

is referred to as the system of discretized equations derived by the implicit midpoint rule. The constant $h > 0$ is the step size for time, and $\boldsymbol{z}^{(n)}$ is the approximation of $\boldsymbol{z}(nh)$.

(2) Prove that the system of discretized equations derived from $(*)$ by the implicit midpoint rule also has L and K shown in (1) as its invariants.

(3) Suppose that $(**)$ has a quadratic invariant of the form $\boldsymbol{z}^\top C\boldsymbol{z}$, where C is a symmetric matrix. Show that the system of discretized equations derived by the implicit midpoint rule also has $\boldsymbol{z}^\top C\boldsymbol{z}$ as its invariant.

† The system of equations $(*)$ describes the rotation of a free rigid body in the three dimensional space, and the constants I_1, I_2, and I_3 are the principal moments of inertia.

Problem 2

Let T be a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$ represented as

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \quad \text{i.e.,} \quad T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

A point $\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$ is called a fixed point if $T \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$. Let

$$J(\overline{x}, \overline{y}) = \begin{pmatrix} \dfrac{\partial f}{\partial x}(\overline{x}, \overline{y}) & \dfrac{\partial f}{\partial y}(\overline{x}, \overline{y}) \\ \dfrac{\partial g}{\partial x}(\overline{x}, \overline{y}) & \dfrac{\partial g}{\partial y}(\overline{x}, \overline{y}) \end{pmatrix}$$

denote the Jacobian matrix of T at $\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$. If each eigenvalue of $J(\overline{x}, \overline{y})$ has an absolute value less than one, the fixed point $\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$ is said to be asymptotically stable. The fixed point is called unstable if at least one of the eigenvalues is larger than one in absolute value.

(1) Consider the plane with the determinant of $J(\overline{x}, \overline{y})$, $\det J(\overline{x}, \overline{y})$, as the horizontal axis and the trace of $J(\overline{x}, \overline{y})$, $\operatorname{tr} J(\overline{x}, \overline{y})$, as the vertical axis. Sketch the region where the fixed point $\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$ is asymptotically stable.

(2) The map T with $f(x,y) = -ax^2 + y + 1$ and $g(x,y) = bx$ is called the Hénon map, where a and b are nonzero real parameters and $|b| < 1$. Find fixed points of the Hénon map. Examine the stabilities of the fixed points and show the result on the parameter plane with b as the horizontal axis and a as the vertical axis.

(3) Let the Hénon map T be represented as a composition of a nonlinear map T_1, a linear map $T_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, and a linear map $T_3 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ as follows:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = T_3\left(T_2\left(T_1\begin{pmatrix} x \\ y \end{pmatrix}\right)\right).$$

Give the concrete form of $T_1\begin{pmatrix} x \\ y \end{pmatrix}$. Furthermore, let $a = 1.4$ and $b = 0.3$, and consider the rectangle region $A = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; -1 \leq x \leq 1,\ \ -0.1 \leq y \leq 0.1\right\}$. Sketch the regions $T_1(A)$, $T_2(T_1(A))$, and $T_3(T_2(T_1(A)))$ and calculate their areas.

(4) Show that the Hénon map T is one-to-one from $\mathbb{R}^2$ onto $\mathbb{R}^2$, namely, bijective.

† A discrete-time dynamical system is defined by the map T as follows:

$$x_{n+1} = f(x_n, y_n),$$
$$y_{n+1} = g(x_n, y_n),$$

where n denotes the discrete time. The concepts of fixed points and stabilities discussed above are related to this dynamical system.

Problem 3

Suppose that the weight μ of a body is measured twice by an experiment and that X_1 and X_2 are the observed values. Their sample mean is denoted by $\overline{X} = (X_1 + X_2)/2$. The measurement errors $X_1 - \mu$ and $X_2 - \mu$ are independently distributed according to the normal distribution with mean 0 and variance 1.

(1) Let $Y = (X_1 - X_2)/2$. Obtain the simultaneous probability distribution of $\overline{X}$ and Y.

(2) Obtain the conditional expectation $\mathrm{E}[X_1|\overline{X}]$ of X_1 given the sample mean $\overline{X}$. What are the expected value and variance of $\mathrm{E}[X_1|\overline{X}]$?

(3) Let $f(X_1, X_2)$ be a function of X_1 and X_2. Show that the conditional expectation $\mathrm{E}[f(X_1, X_2)|\overline{X}]$ of $f(X_1, X_2)$ given $\overline{X}$ is a function of $\overline{X}$, independent of μ.

(4) Suppose that $f(X_1, X_2)$ is an unbiased estimator of μ. Show that $\mathrm{E}[f(X_1, X_2)|\overline{X}]$ is an unbiased estimator of μ and that the variance of $\mathrm{E}[f(X_1, X_2)|\overline{X}]$ is not greater than the variance of $f(X_1, X_2)$.

(5) Explain briefly the concept of sufficient statistics. What properties do sufficient statistics have?

Problem 4

Suppose that we are given n-dimensional vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k \in \mathbb{R}^n$ and the set of their indices $K = \{1, 2, \ldots, k\}$. For a subset $L \subseteq K$, we define

$$C(L) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}_i^\top \boldsymbol{x} < 0 \ \ (\forall i \in L), \quad \boldsymbol{a}_j^\top \boldsymbol{x} > 0 \ \ (\forall j \in K \setminus L) \right\},$$

where $\boldsymbol{a}_i^\top$ stands for the transpose of $\boldsymbol{a}_i$ and $K \setminus L$ means the set of elements in K and not in L.

(1) Let $k = 3$, $n = 2$, $\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{a}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\boldsymbol{a}_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Obtain all subsets L of $K = \{1, 2, 3\}$ such that $C(L)$ is nonempty.

(2) Assume that $n = 3$. We denote by $\ell(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k)$ the number of L such that $C(L)$ is nonempty and by $r_k = \max_{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k} \ell(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k)$ the maximum of $\ell(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k)$ over $(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k)$. Obtain r_1, r_2, r_3, r_4, and r_5. Express r_k as a function of k.

(3) Prove that, if $C(L)$ is nonempty for each $L \subseteq K$, then the set $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k\}$ is linearly independent.

(4) Prove the converse, i.e., that, if the set $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k\}$ is linearly independent, then $C(L)$ is nonempty for each $L \subseteq K$.

Problem 5

Suppose that m and n are positive integers, and that m is much smaller† than n. Let M be a set of m symbols. Let a be a one-dimensional array of size n. A symbol of M is assigned to each element $a[i]$ ($i = 1, 2, \ldots, n$) of the array a. In other words, $a[i] \in M$ for any $i = 1, 2, \ldots, n$. The array a contains all of the symbols in M in such a way that if a symbol is assigned to two elements of a, that symbol is also assigned to all the inbetween elements of a. In other words, if $a[i] = a[j]$ for some $i < j$, then $a[i] = a[k]$ for all k such that $i + 1 \leq k \leq j - 1$. Integer i is called a *change position* if $a[i] \neq a[i + 1]$. There are exactly $m - 1$ change positions. Design an algorithm for extracting all the change positions in a with time complexity smaller than $\mathrm{O}(n)$, and explain its behavior together with its correctness. Also, evaluate its time complexity‡

† The expression "m is much smaller than n" means that $m = \mathrm{O}(n^{\alpha})$ holds for some $\alpha < 1$.

‡ The symbols are already stored in the array a, and hence you need not consider the time to store the symbols in the array. Assume that the symbol in each element $a[i]$ can be retrieved in a constant time independent of n.